Returns Based Style Analysis

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This is a short note about the implementation of Returns Based Style Analysis (RBSA) a technique first described in [Sharpe, 1992]. It is a method to generate a replicating factor portfolio to match a portfolio return stream.

RBSA starts from a simple multi-factor model

$$R = \alpha + F\,w$$

For n periods of returns and k factors,
R is the n x 1 vector of returns to the portfolio
F is the n x k matrix of returns to the factors
α is the n x 1 vector of excess returns (can be positive or negative)
w is the k x 1 vector of factor weights

In contrast to a multiple-regression analysis where the goal is to minimize the value of α, in RBSA we are minimizing the variance of α. In this way we are also looking for $\bar{\alpha}$ itself to be an output of the analysis. α captures any return which is not dependent on the factors specified. If our factors span the portfolio, then α must be the managers value added return. Note that the manager's value added return can be positive or negative.

Usually with RBSA we apply some constraints to the problem which require the use of quadratic optimization techniques. The most common constraints are that we want the replicating portfolio to be fully invested in the factors (budget constraint) and we do not want short exposures to the factors. We use these constraints because the factors are typically investable. Thus, the optimization problem to be solved for RBSA is

$$min_w(var(\alpha))$$

$$\text{s.t. } \sum_{i=1}^{T} w_i = 1 \text{ and } w_i \geq 0 \; \forall \; i$$

In order to solve this problem, we need to derive an expression for var(α) in terms of our factor model.

$$var(\alpha) = var(R - F\,w)$$

From the definition of variance

$$var(\alpha) = \sum_{i=1}^{T} \frac{(\alpha_i - \bar{\alpha})^2}{T}$$

Expanding $\bar{\alpha}$

$$var(\alpha)=\frac{1}{T}\left[\sum_{i=1}^{T}(R_i-F_i w)^2-\left(\frac{\sum_{i=1}^{T}(R_i-F_i w)}{T}\right)^2\right]$$

Expanding out all the terms

$$var(\alpha)=\frac{1}{T}\left[(R^T R-R^T F w-(F w)^T R+w^T F^T F w)-\frac{((e^T R)^2-e^T R e^T F w-R^T e(F w)^T e+(e^T F w)^2)}{T}\right]$$

Where e is a n x 1 vector of 1's.

Combining terms, using the relationship $e^T R=R^T e$ for n x 1 vectors R and e, then we can create a quadratic expression in w

$$var(\alpha)=\left(\frac{R^T R}{T}-\frac{(e^T R)^2}{T^2}\right)-2\left(\frac{R^T F}{T}-(\frac{e^T R}{T^2})e^T F\right)w+w^T\left((\frac{F^T F}{T}-\frac{1}{T^2}F^T e e^T F)\right)w$$

If we define the term M as

$$M=\left(I-\frac{e e^T}{T}\right)$$

We can simplify the formula as shown

$$var(\alpha)=\frac{1}{T}\left(R^T M R-2 R^T M F w+w^T F^T M F w\right)$$

If we let

$$Q=\frac{2}{T}F^T M F$$
$$g=-\frac{2}{T}R^T M F$$

We can drop the constant term and rewrite the problem as the following quadratic in w.

$$min_w(\frac{1}{2}w^T Q w+g^T w)$$
$$\text{s.t. }\sum_{i=1}^{T}w_i=1 \text{ and } w_i\geq 0\,\forall\, i$$

This is the standard form to be solved for *w* using quadratic optimization techniques. Solving this problem will give us the set of factor weights which minimize the variance of α.

References

[Sharpe 1992]. Asset Allocation: Management Style and Performance Measurement, William F. Sharpe, Journal of Portfolio Management, Winter 1992, pp. 7-19. (Available at http://www.stanford.edu/~wfsharpe/art/sa/sa.htm)

Optimization Methods in Finance, Gérard Cornuéjols and Reha Tütüncü, Cambridge University Press, 2006.